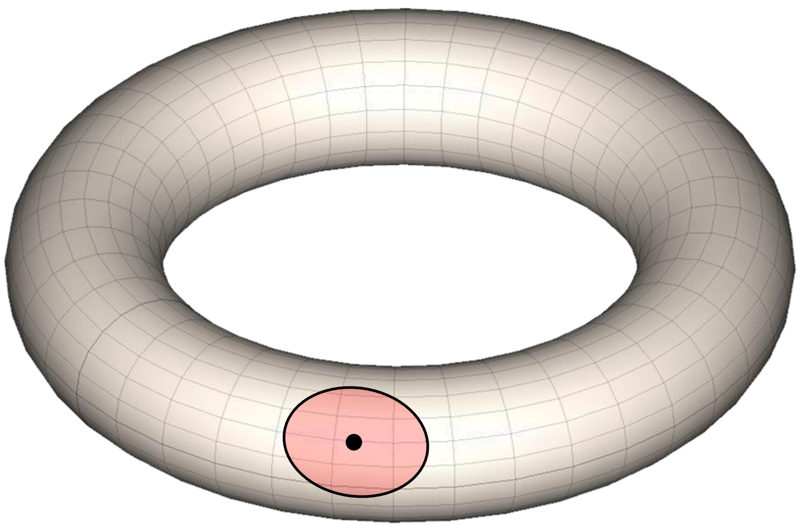
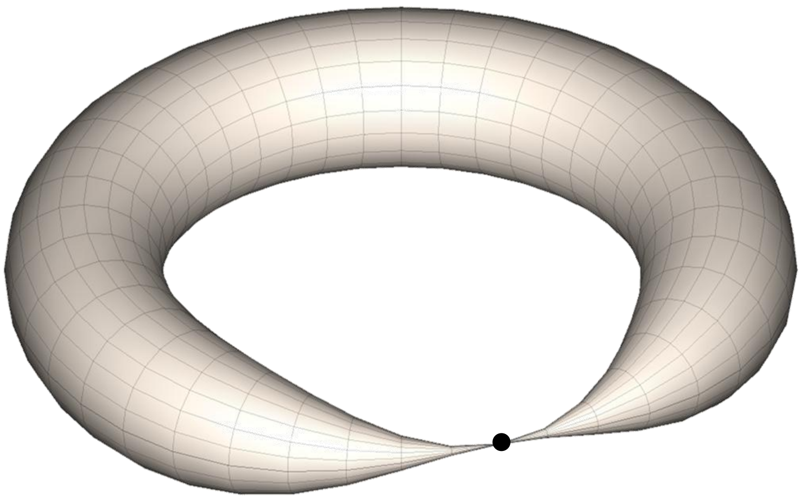
Differential geometry I

Manifolds

A topological space in which every point has a neighborhood homeomorphic to $\mathbb{R}^n$ (**topological disc**) is called an *n*-**dimensional** (or *n*-) **manifold**



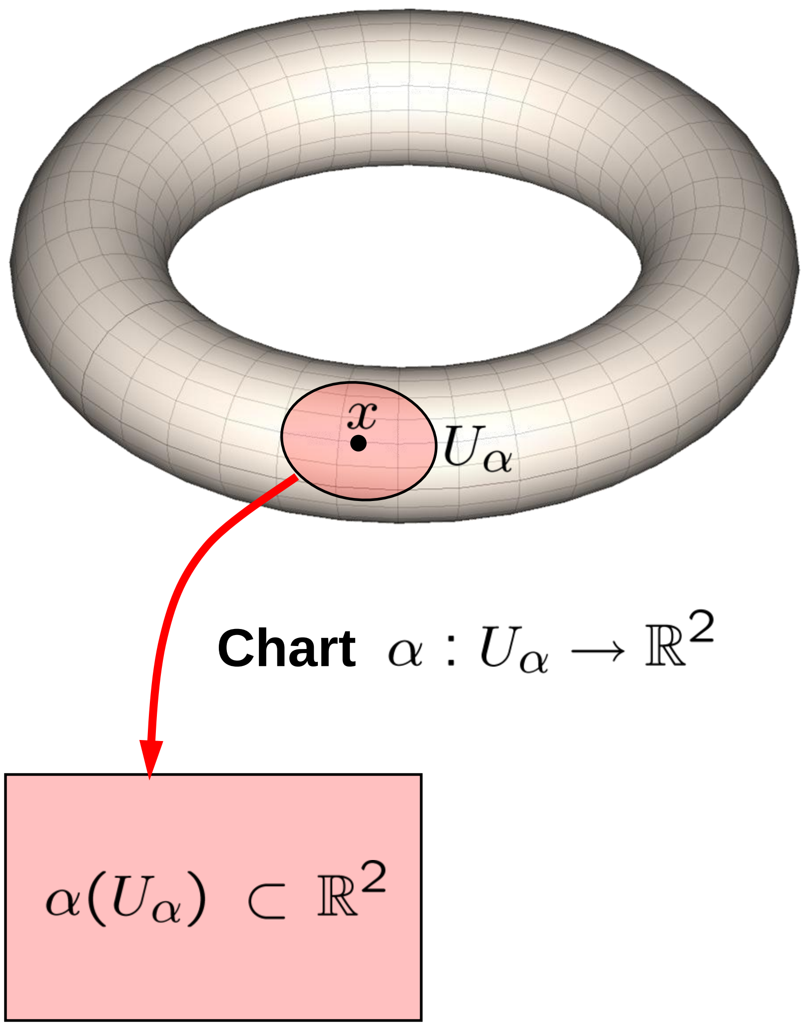
2-manifold



Not a manifold

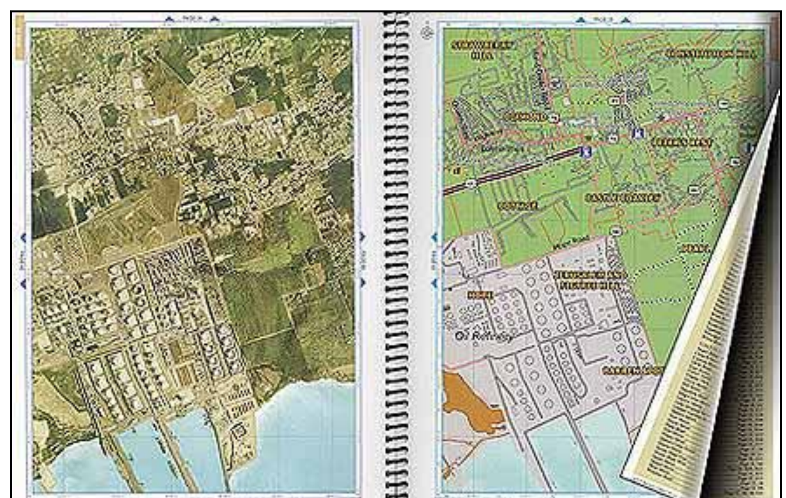
Earth is an example of a 2-manifold

Charts and atlases

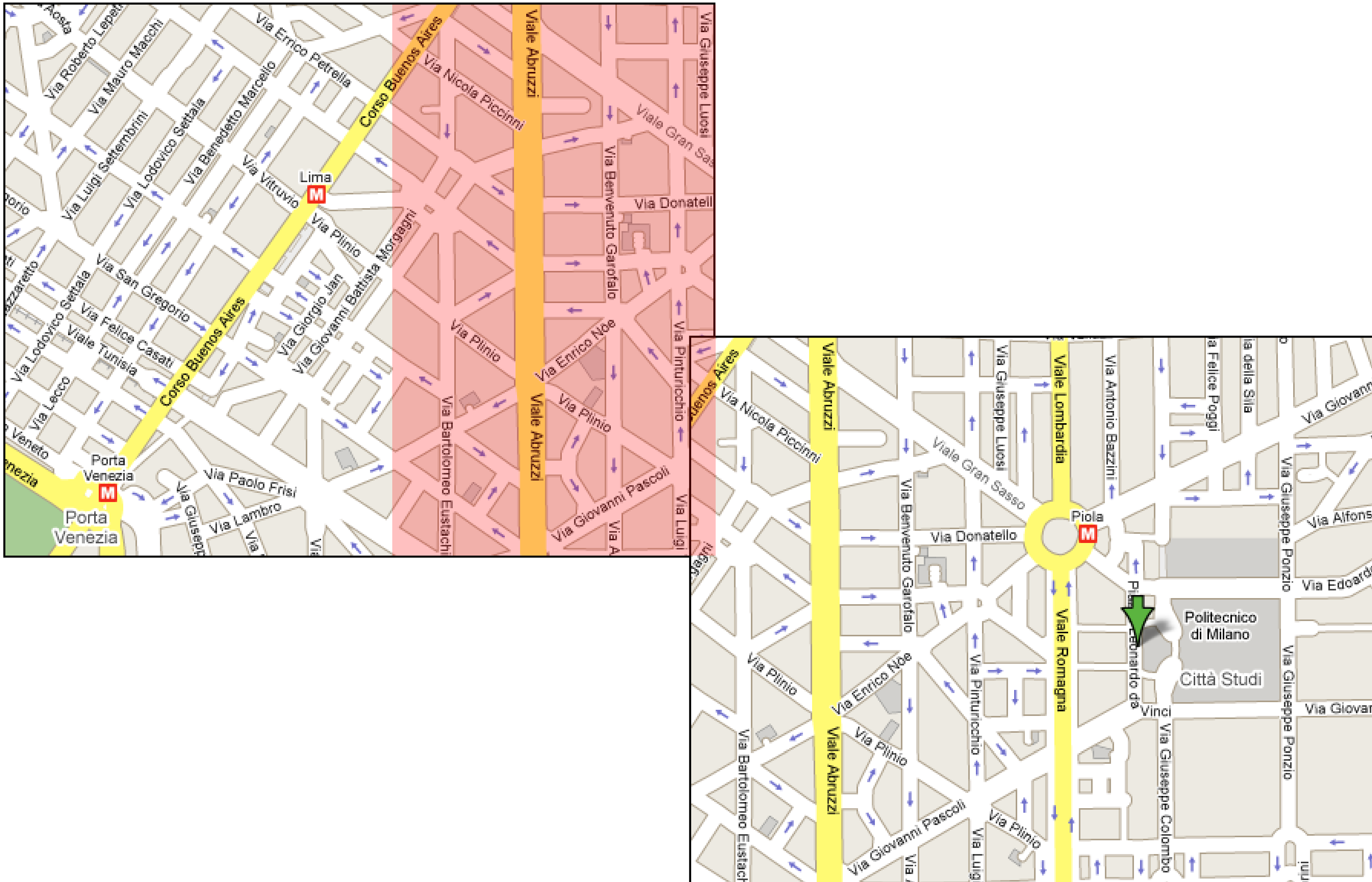


A homeomorphism $\alpha : U_\alpha \to \mathbb{R}^n$ from a neighborhood U_α of $x \in X$ to $\mathbb{R}^n$ is called a **chart**

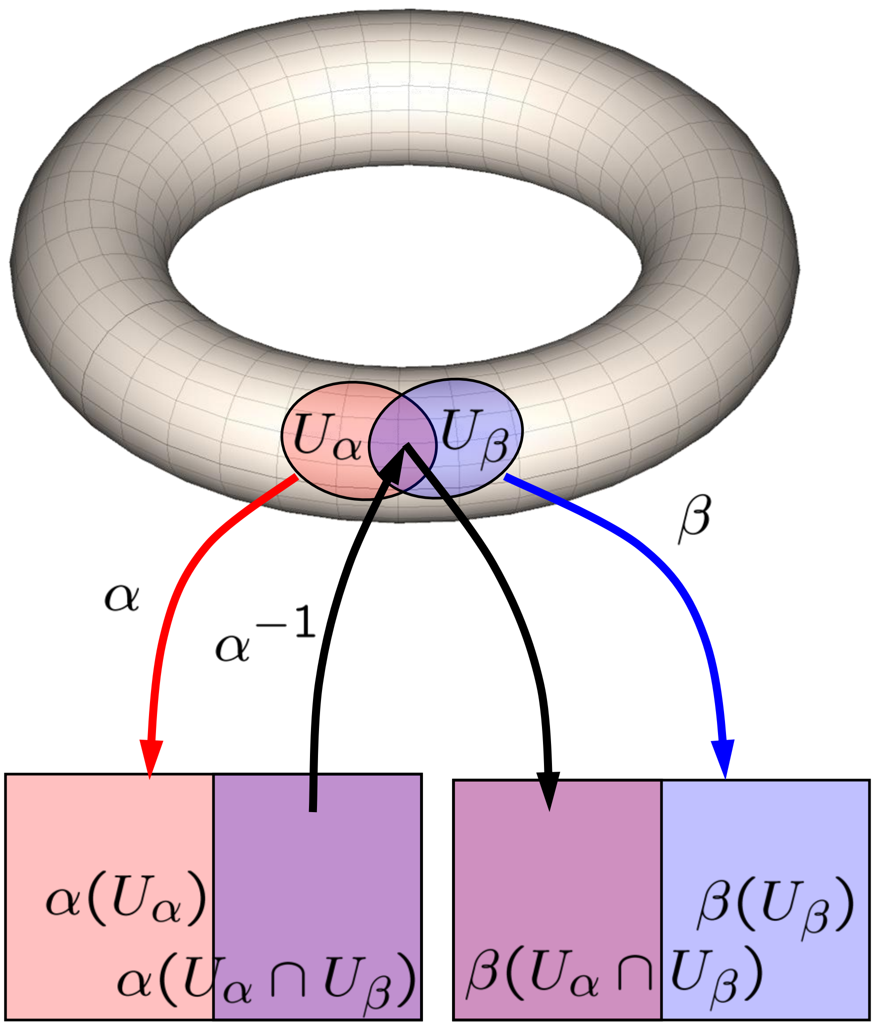
A collection of charts whose domains cover the manifold is called an **atlas**



Charts and atlases



Smooth manifolds



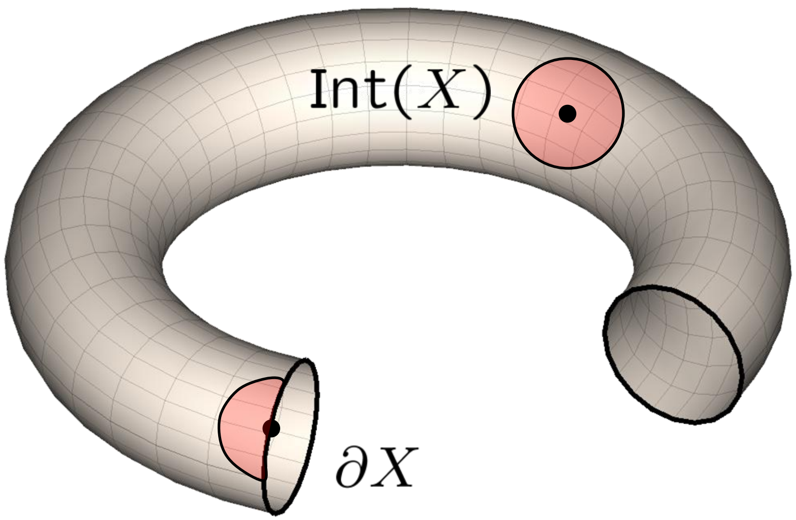
Given two charts $\alpha : U_\alpha \rightarrow \mathbb{R}^n$ and $\beta : U_\beta \rightarrow \mathbb{R}^n$ with overlapping domains $U_\alpha \cap U_\beta$ change of coordinates is done by **transition function**

$$\beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \rightarrow \mathbb{R}^n$$

If all transition functions are $\mathcal{C}^r$, the manifold is said to be $\mathcal{C}^r$

A $\mathcal{C}^\infty$ manifold is called **smooth**

Manifolds with boundary



A topological space in which every point has an open neighborhood homeomorphic to either

- topological disc $\mathbb{R}^n$; or
- topological half-disc $[0, \infty) \times \mathbb{R}^{n-1}$

is called a **manifold with boundary**

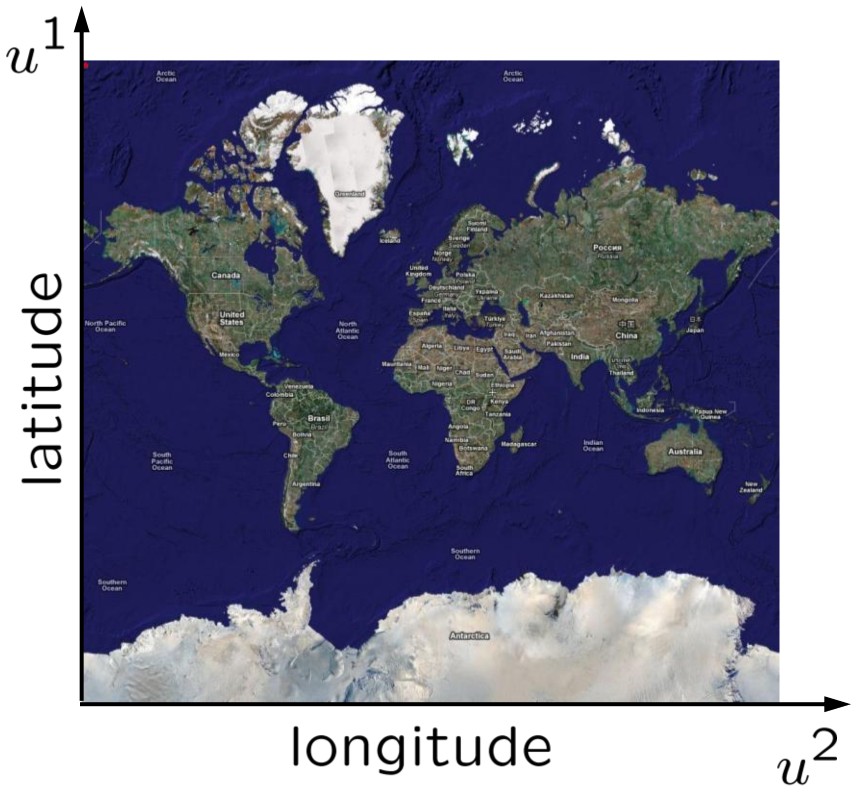
Points with disc-like neighborhood are called **interior**, denoted by $\text{Int}(X)$

Points with half-disc-like neighborhood are called **boundary**, denoted by ∂X

Embedded surfaces

- Boundaries of **tangible physical objects** are two-dimensional **manifolds**.
- They reside in (are **embedded** into, are **subspaces** of) the **ambient** three-dimensional **Euclidean space**.
- Such manifolds are called **embedded surfaces** (or simply **surfaces**).
- Can often be described by the map $x : U \subset \mathbb{R}^2 \rightarrow X \subset \mathbb{R}^3$
  - $U \subset \mathbb{R}^2$ is a **parametrization domain**.
  - the map $x(u) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))$ is a **global parametrization** (**embedding**) of X .
- Smooth global parametrization **does not always exist** or is easy to find.
- Sometimes it is more convenient to work with **multiple charts**.

Parametrization of the Earth



$$U = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [-\pi, \pi]$$

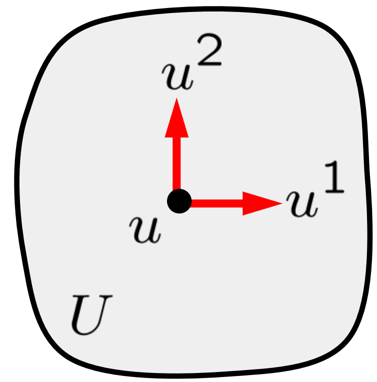
$$\begin{aligned} x^1 &= r \cos u^2 \cos u^1 \\ x^2 &= r \sin u^2 \cos u^1 \\ x^3 &= r \sin u^1 \end{aligned}$$

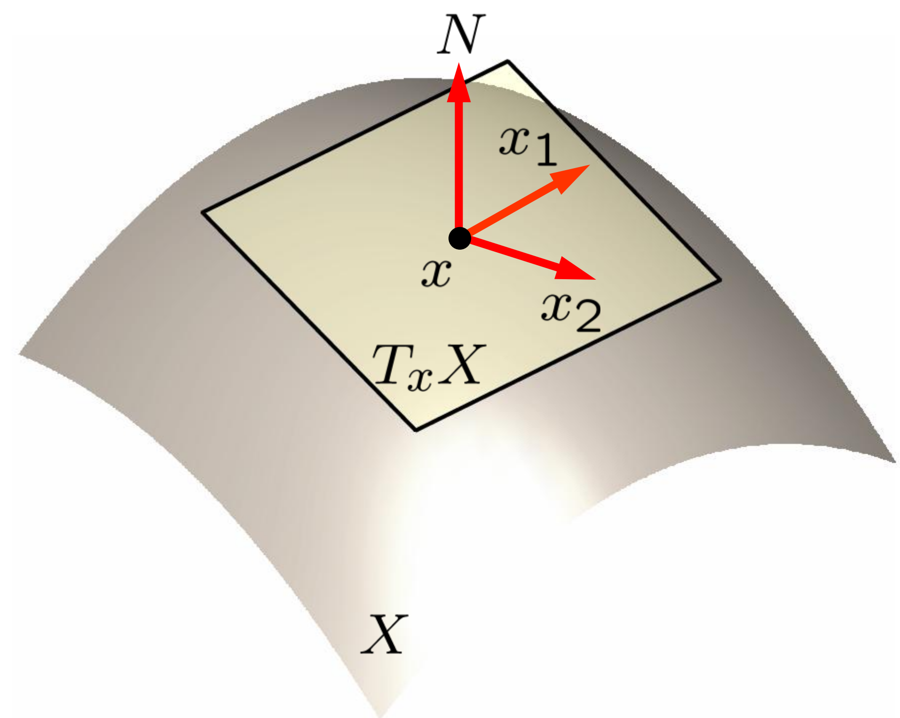
Tangent plane & normal

- At each point $u \in U$, we define **local system of coordinates**

$$x_1 = \frac{\partial x}{\partial u^1} \qquad x_2 = \frac{\partial x}{\partial u^2}$$

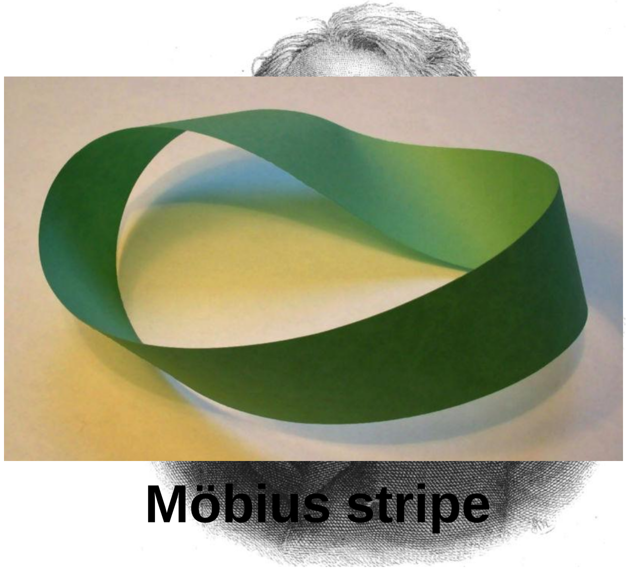
- A parametrization is **regular** if x_1 and x_2 are **linearly independent**.
- The plane $T_x X = \mathrm{span}\{x_1, x_2\}$ is **tangent plane** at $x = x(u)$.
- **Local Euclidean approximation** of the surface.
- $N \perp T_x X$ is the **normal** to surface.





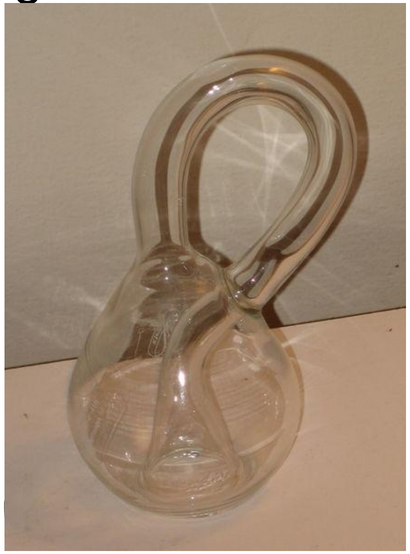
Orientability

- Normal is defined up to a **sign**.
- Partitions ambient space into **inside** and **outside**.
- A surface is **orientable**, if normal N depends smoothly on x .



Möbius stripe

August Ferdinand Möbius



Felix Christian Klein
(1849-1925)

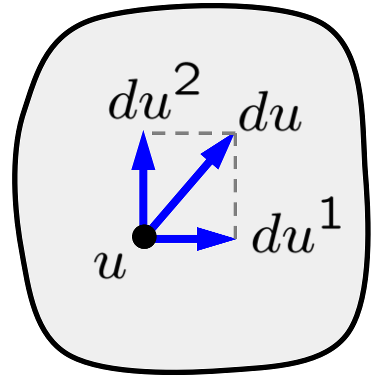
Klein bottle
(3D section)

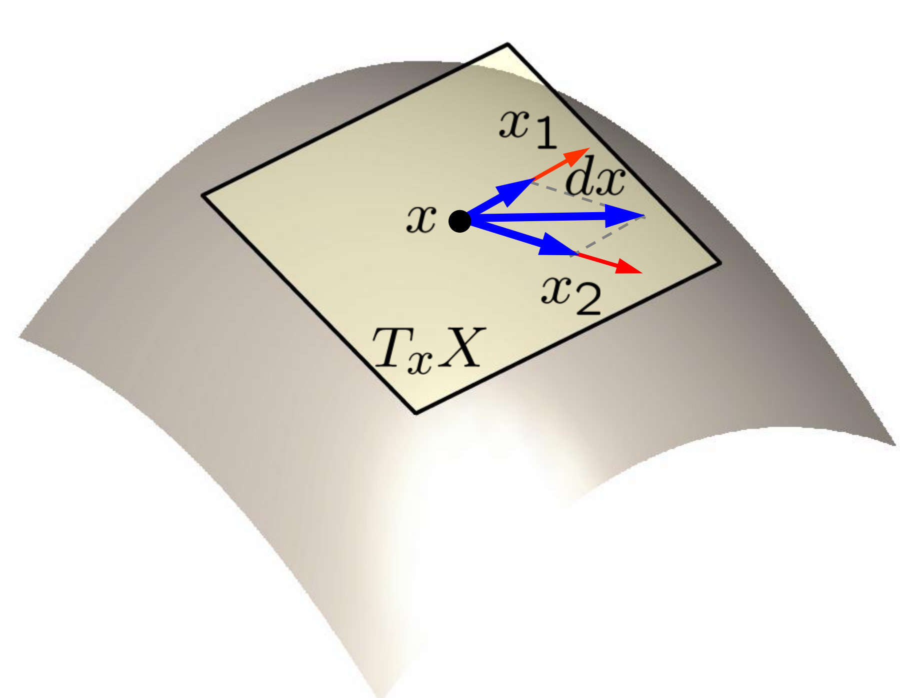
First fundamental form

- **Infinitesimal displacement** on the chart du.
- Displaces x **on the surface** by

$$\begin{aligned} dx &= x(u+du) - x(u) \\ &= x_1 du^1 + x_2 du^2 \\ &= J du \end{aligned}$$

- J is the **Jacobain matrix**, whose columns are x_1 and x_2.

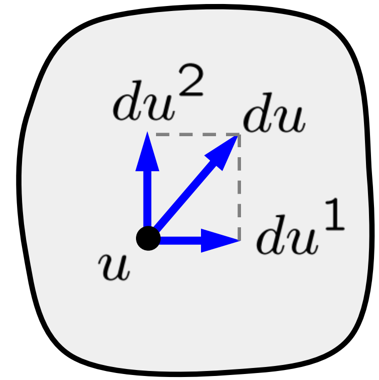




First fundamental form

- **Length** of the displacement

$$d\ell^2 = \|dx\|^2 = du^\top J^\top J du$$
$$= du^\top G du$$



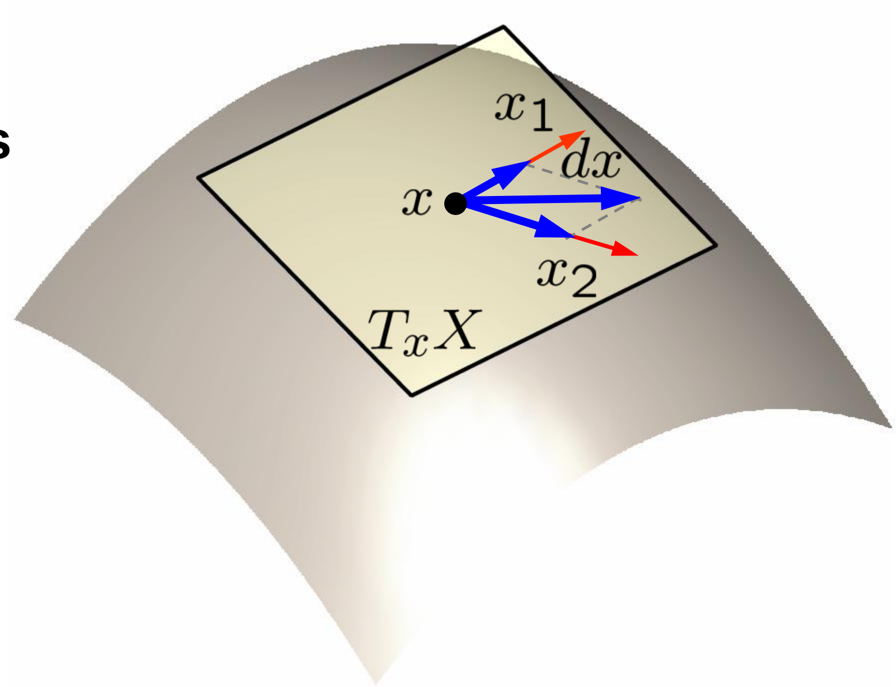
- G is a **symmetric positive definite** 2×2 matrix.
- Elements of G are **inner products**

$$g_{ij} = \langle x_i, x_j \rangle$$

- Quadratic form

$$d\ell^2 = du^\top G du$$

is the **first fundamental form**.



First fundamental form of the Earth

- **Parametrization**

$$x = (r\cos u^2 \cos u^1, r\sin u^2 \cos u^1, r\sin u^1)$$

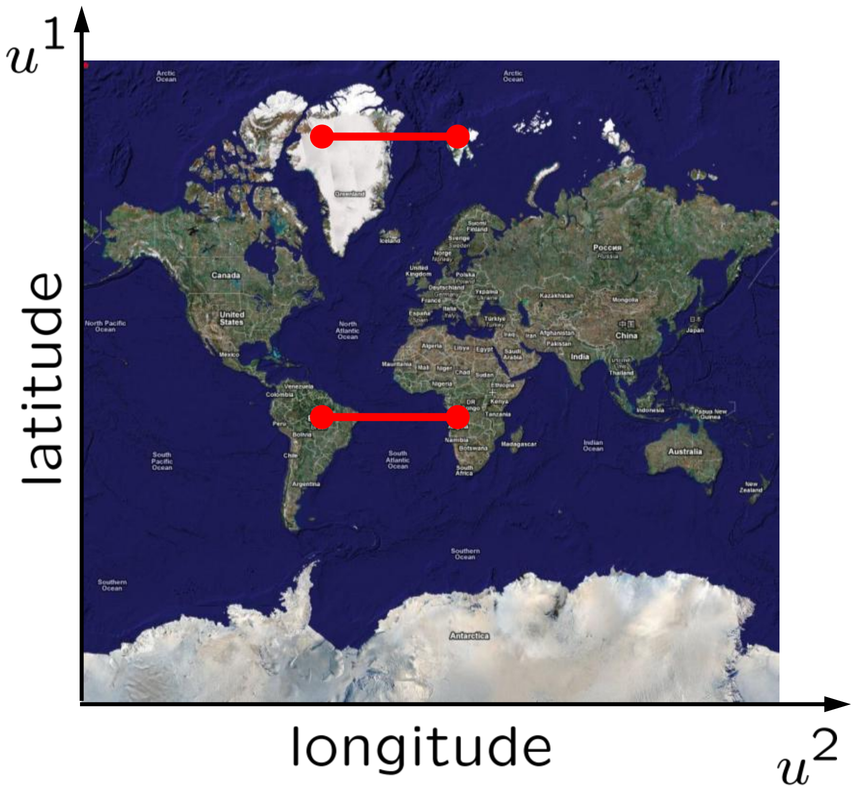
- **Jacobian**

$$\begin{aligned} x_1 &= (-r\cos u^2 \sin u^1, -r\sin u^2 \sin u^1, r\cos u^1) \\ x_2 &= (-r\sin u^2 \cos u^1, r\cos u^2 \cos u^1, 0) \end{aligned}$$

- **First fundamental form**

$$\begin{aligned} G &= \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle \\ \langle x_1, x_2 \rangle & \langle x_2, x_2 \rangle \end{pmatrix} \\ &= r \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix} \end{aligned}$$

First fundamental form of the Earth



$$G = r\begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix}$$

First fundamental form

- Smooth **curve** on the chart:

$$\gamma : [a,b] \to U$$

- Its image on the surface:

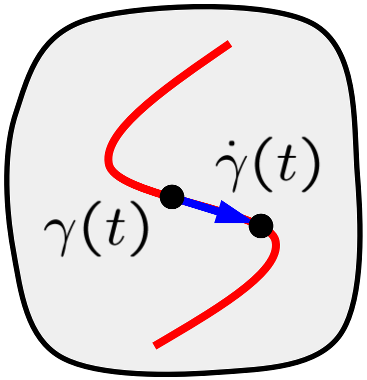
$$\Gamma = x \circ \gamma$$

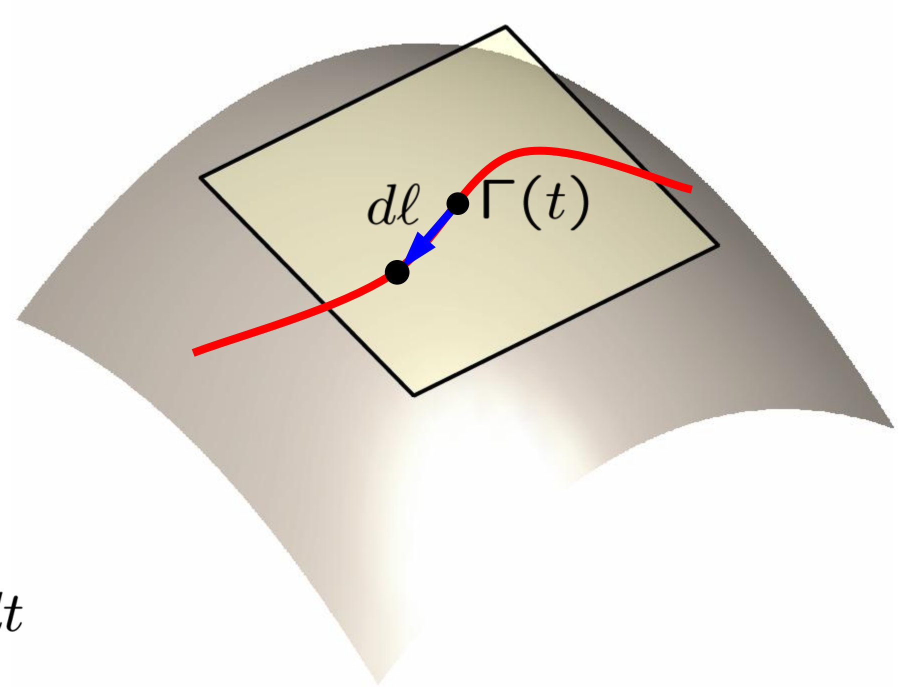
- Displacement on the curve: dt
- Displacement in the chart:

$$\begin{aligned} d\gamma &= \gamma(t+dt) - \gamma(t) \\ &= \dot{\gamma}(t)dt \end{aligned}$$

- **Length** of displacement on the surface:

$$d\ell = \sqrt{\dot{\gamma}(t)^{\top} G(\gamma(t)) \dot{\gamma}(t)}\, dt$$





Intrinsic geometry

- **Length** of the curve

$$L(\Gamma) = \int_\Gamma d\ell = \int_a^b \sqrt{\dot\gamma(t)^\top G(\gamma(t))\dot\gamma(t)}\,dt$$

- First fundamental form induces a **length metric** (**intrinsic metric**)

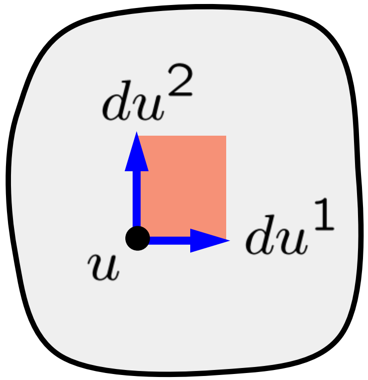
$$d_X(x_1, x_2) = \min_{\substack{\Gamma \\ \Gamma(0)=x_1, \Gamma(1)=x_2}} L(\Gamma)$$

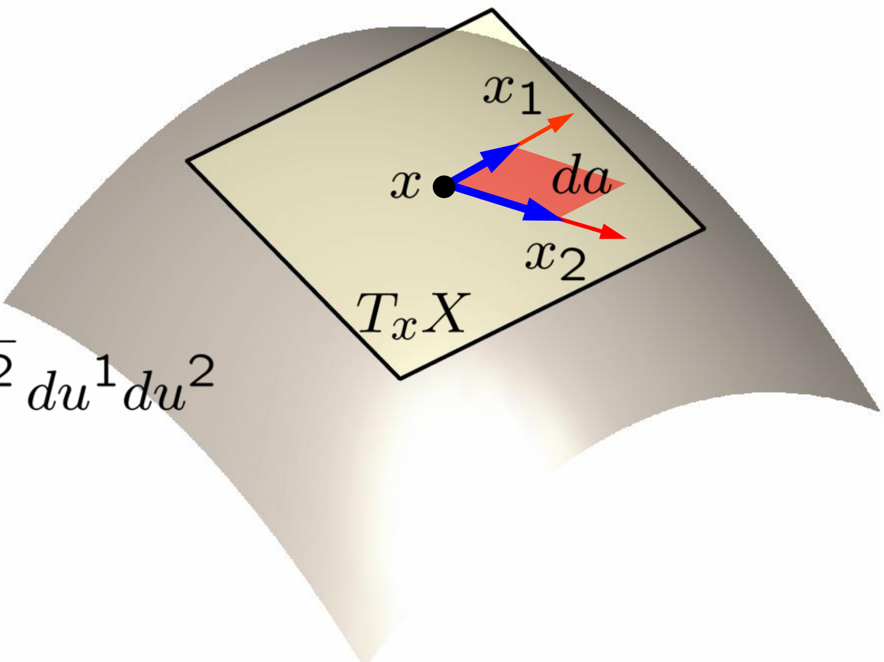
- **Intrinsic geometry** of the shape is **completely described** by the first fundamental form.
- First fundamental form is **invariant to isometries**.

Area

- **Differential area element** on the chart: **rectangle** $du^1 \times du^2$
- Copied by x to a **parallelogram** $du^1 x_1 \times du^2 x_2$ in **tangent space**.
- Differential area element **on the surface**:

$$
\begin{aligned}
da &= \|du^1 x_1 \times du^2 x_2\| \\
&= \|x_1 \times x_2\| du^1 du^2 \\
&= \sqrt{\|x_1\|^2 \|x_2\|^2 - \langle x_1, x_2 \rangle^2}\, du^1 du^2 \\
&= \sqrt{g_{11} g_{22} - g_{12}^2}\, du^1 du^2 \\
&= \sqrt{\det G}\, du^1 du^2
\end{aligned}
$$





Area

- **Area** or a region $\Omega \subseteq X$ charted as $\Omega = x(\omega \subseteq U)$

$$\mu(\Omega) = \int_\Omega da = \int_\omega \sqrt{\det G} du^1 du^2$$

- **Relative area**

$$\nu(\Omega) = \frac{\mu(\Omega)}{\mu(X)}$$

- **Probability** of a point on X picked at **random** (with uniform distribution) to fall into Ω .

  Formally

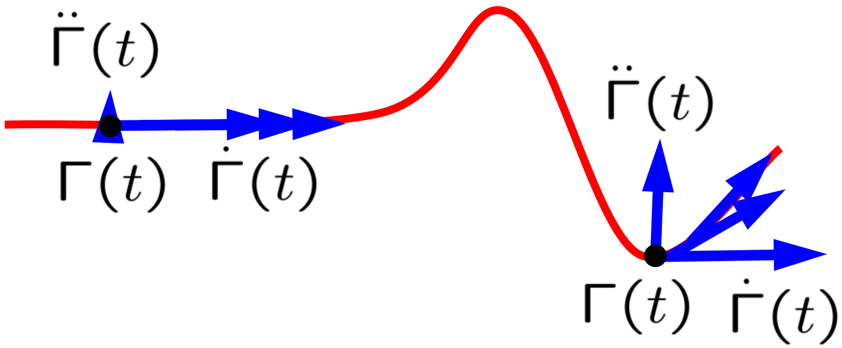
- $\mu(\Omega), \nu(\Omega)$ are **measures** on X .

Curvature in a plane

- Let $\Gamma : [0, L] \to \mathbb{R}^2$ be a **smooth curve** parameterized by **arclength**

$$\int_a^b \|\dot{\Gamma}(t)\| dt = |a - b|$$

- Γ **trajectory** of a race car driving at constant velocity.
- $\dot{\Gamma}$ **velocity** vector (rate of change of position), tangent to path.
- $\ddot{\Gamma}$ **acceleration** (**curvature**) vector, perpendicular to path.
- $\kappa = \|\ddot{\Gamma}\|_2$ **curvature**, measuring rate of rotation of velocity vector.

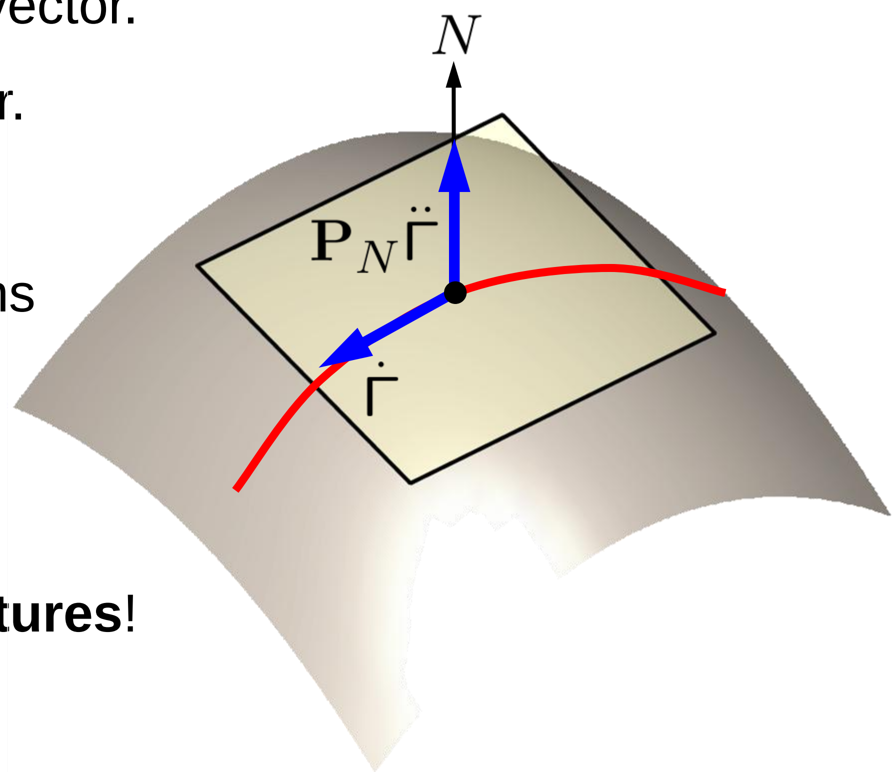


Curvature on surface

- Now the car drives on terrain X.
- Trajectory described by $\Gamma : [0, L] \rightarrow X$.
- **Curvature vector** $\ddot{\Gamma}$ decomposes into
  - $\mathbf{P}_{T_\Gamma X}\ddot{\Gamma}$ **geodesic curvature** vector.
  - $\mathbf{P}_N\ddot{\Gamma}$ **normal curvature** vector.
- **Normal curvature** $\kappa_n = \langle N, \ddot{\Gamma} \rangle$
- Curves passing in different directions have different values of κ_n.

Said differently:

- A point $x \in X$ has **multiple curvatures**!



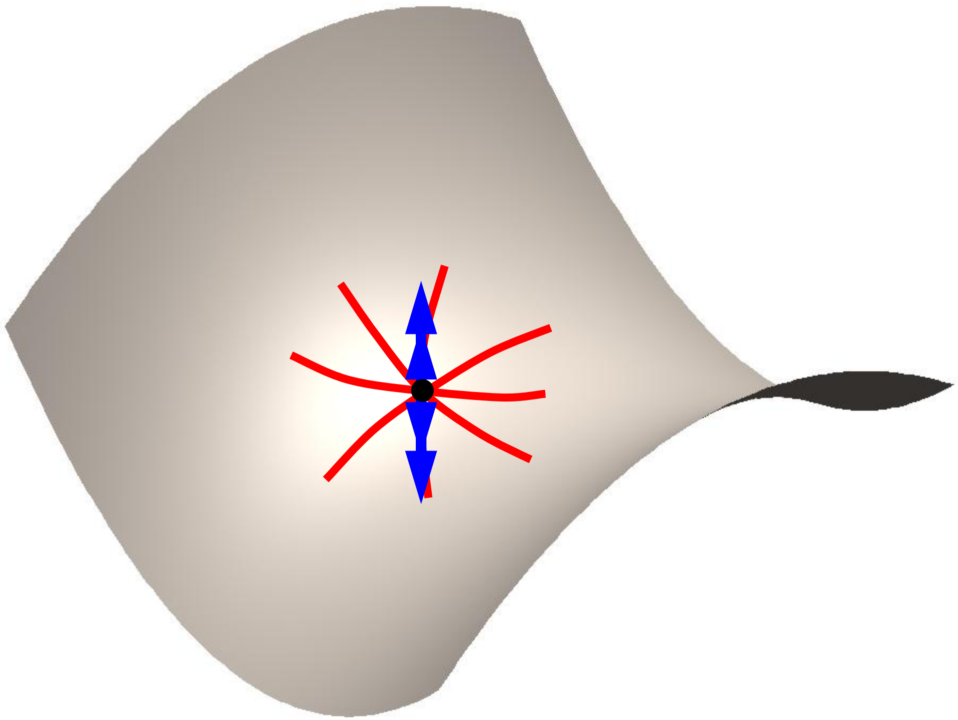
Principal curvatures

- For each direction $v \in T_x X$, a curve Γ passing through $\Gamma(0) = x$ in the direction $\dot{\Gamma}(0) = v$ may have a different normal curvature κ_n
- **Principal curvatures**

$$\kappa_1 = \min_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle$$
$$\kappa_2 = \max_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle$$

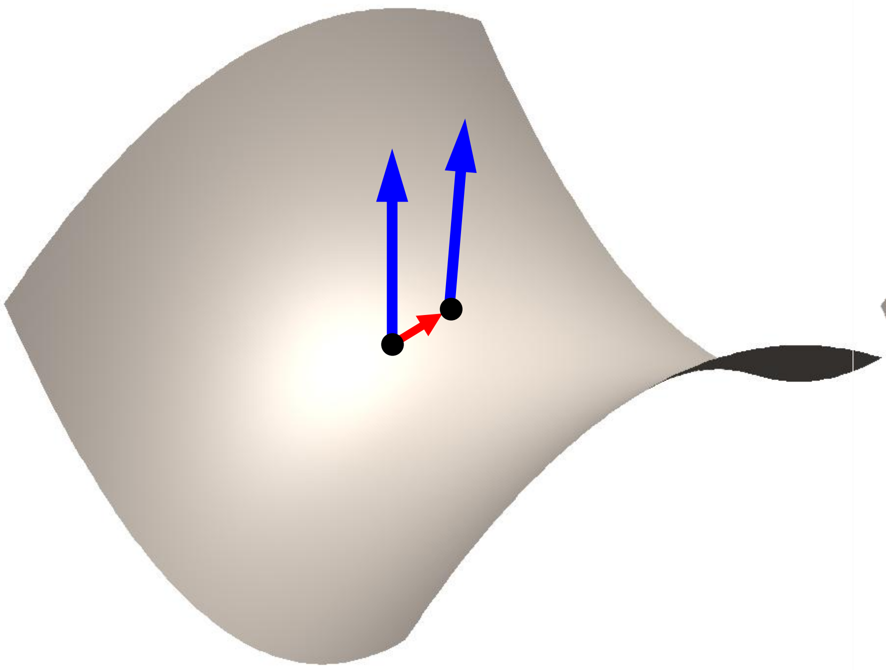
- **Principal directions**

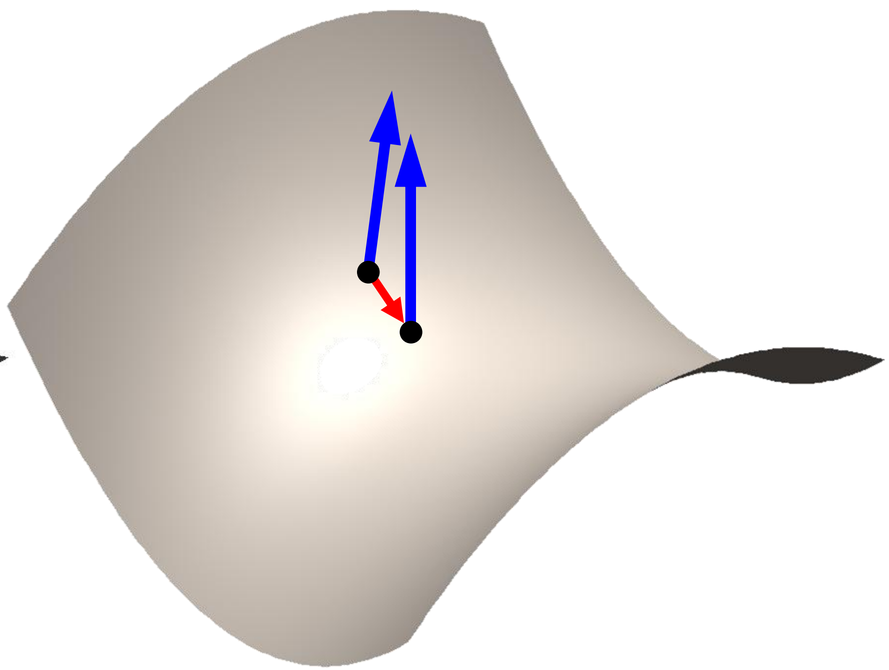
$$T_1 = \arg \min_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle$$
$$T_2 = \arg \max_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle$$



Curvature

- **Sign of normal curvature** = direction of rotation of normal to surface.
  - $\kappa_n > 0$ a step in direction $\dot{\Gamma}$ rotates N in **same direction**.
  - $\kappa_n < 0$ a step in direction $\dot{\Gamma}$ rotates N in **opposite direction**.





Curvature: a different view

- A **plane** has a **constant normal** vector, e.g. $N = (0, 0, 1)$.
- We want to quantify how a curved surface is different from a plane.
- Rate of change of N i.e., **how fast the normal rotates**.

- **Directional derivative** of N at point $x \in X$ in the direction $v \in T_x X$

$$D_v N = \lim_{t \to 0} \frac{1}{t}(N(\Gamma(t)) - N(x)) = \frac{d}{dt} N(\Gamma(t))\Big|_{t=0}$$

$\Gamma : (-\epsilon, +\epsilon) \to X$ is an arbitrary smooth curve with $\Gamma(0) = x$ and $\dot{\Gamma}(0) = v$.

Curvature

- $D_v N$ is a vector in $\mathbb{R}^3$ measuring the change in N as we make differential steps in the direction v.
- Differentiate $1 = \langle N, N \rangle$ w.r.t. t

$$0 = \frac{d}{dt}\langle N, N \rangle = 2\langle D_v N, N \rangle$$

- Hence $D_v N \perp N$ or $D_v N \in T_x X$.
- **Shape operator** (a.k.a. **Weingarten map**): is the map $S : T_x X \to T_x X$ defined by

$$S(v) = -D_v N$$

Julius Weingarten
(1836-1910)

Shape operator

- Can be expressed in **parametrization coordinates** as $S(v) = Sv$

  S is a 2×2 matrix satisfying

$$\begin{pmatrix} S(x_1) \\ S(x_2) \end{pmatrix} = S \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Multiply by (x_1, x_2)

$$\begin{pmatrix} S(x_1) \\ S(x_2) \end{pmatrix} (x_1, x_2) = S \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (x_1, x_2)$$

$$B = SG$$

where

$$B = \begin{pmatrix} \langle S(x_1), x_1 \rangle & \langle S(x_1), x_2 \rangle \\ \langle S(x_2), x_1 \rangle & \langle S(x_2), x_2 \rangle \end{pmatrix} = - \begin{pmatrix} \langle \partial_{u^1} N, x_1 \rangle & \langle \partial_{u^1} N, x_2 \rangle \\ \langle \partial_{u^2} N, x_1 \rangle & \langle \partial_{u^2} N, x_2 \rangle \end{pmatrix}$$

Second fundamental form

- The matrix B gives rise to the **quadratic form**

$$B(v,w) = \langle S(v), w \rangle = w^{\top} B v$$

called the **second fundamental form**.

- Related to **shape operator** and **first fundamental form** by identity

$$S = BG^{-1}$$

Principal curvatures encore

- Let $\Gamma : [0, L] \to X$ be a curve on the surface.
- Since $\dot{\Gamma} \in T_x X$, $\langle \dot{\Gamma}, N \rangle = 0$.
- Differentiate w.r.t. to t

$$0 = \frac{d}{dt}\langle \dot{\Gamma}, N \rangle = \langle \ddot{\Gamma}, N \rangle + \langle \dot{\Gamma}, \frac{d}{dt} N \rangle$$

$$\kappa_n = \langle \ddot{\Gamma}, N \rangle = \langle \dot{\Gamma}, -D_{\dot{\Gamma}} N \rangle = B(\dot{\Gamma}, \dot{\Gamma}) = \dot{\Gamma}^\top B \dot{\Gamma}$$

- $\kappa_1 \leq \dot{\Gamma}^\top B \dot{\Gamma} \leq \kappa_2$
- κ_1 is the **smallest eigenvalue** of B .
- κ_2 is the **largest eigenvalue** of B .
- T_1, T_2 are the corresponding **eigenvectors**.

Second fundamental form of the Earth

- **Parametrization** $x = (r\cos u^2 \cos u^1, r\sin u^2 \cos u^1, r\sin u^1)$
- **Normal**

$$\begin{aligned} N &= (\cos u^2 \cos u^1, \sin u^2 \cos u^1, \sin u^1) \\ \partial_{u^1} N &= (-\cos u^2 \sin u^1, -\sin u^2 \sin u^1, \cos u^1) \\ \partial_{u^2} N &= (-\sin u^2 \cos u^1, \cos u^2 \cos u^1, 0) \end{aligned}$$

- **Second fundamental form**

$$B = -\begin{pmatrix} \langle \partial_{u^1} N, x_1 \rangle & \langle \partial_{u^1} N, x_2 \rangle \\ \langle \partial_{u^2} N, x_1 \rangle & \langle \partial_{u^2} N, x_2 \rangle \end{pmatrix} = -\frac{1}{r} G = \begin{pmatrix} -1 & 0 \\ 0 & -\cos^2 u^1 \end{pmatrix}$$

Shape operator of the Earth

- **First fundamental form**

$$G = r\begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix}$$

- **Second fundamental form**

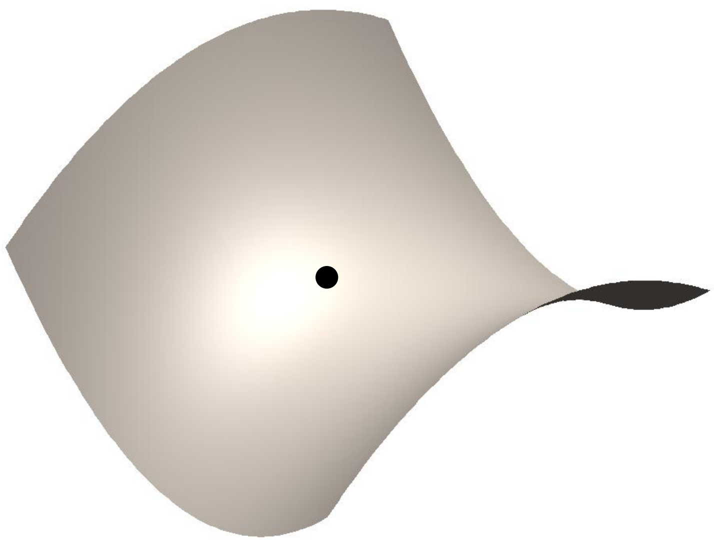
$$B = \begin{pmatrix} -1 & 0 \\ 0 & -\cos^2 u^1 \end{pmatrix}$$

- **Shape operator** $S = BG^{-1} = -\frac{1}{r}I$
- Constant at every point.

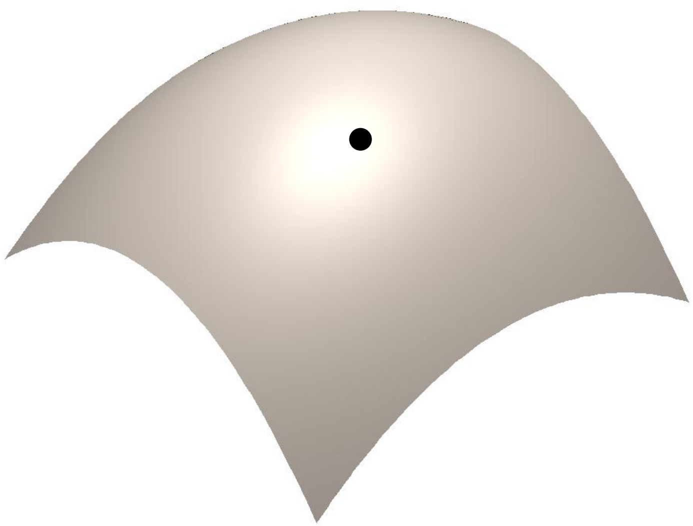
- Is there connection between **algebraic invariants** of shape operator S (trace, determinant) with **geometric invariants** of the shape?

Mean and Gaussian curvatures

- **Mean curvature** $H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\text{trace}\, S$
- **Gaussian curvature** $K = \kappa_1 \kappa_2 = \det\, S$



hyperbolic point $K < 0$



elliptic point $K > 0$

Extrinsic & intrinsic geometry

- **First fundamental** form describes completely the **intrinsic geometry**.
- **Second fundamental** form describes completely the **extrinsic geometry** – the “layout” of the shape in ambient space.
- **First fundamental form** is invariant to **isometry**.
- **Second fundamental form** is invariant to **rigid motion** (**congruence**).
- If X and $f(X)$ are **congruent** (i.e., $f \in \mathrm{Iso}(\mathbb{R}^3)$), then they have identical intrinsic and extrinsic geometries.
- **Fundamental theorem:** a map preserving the first and the second fundamental forms is a congruence.

  Said differently: an isometry preserving second fundamental form is a restriction of Euclidean isometry.